Two-dimensional evolution of surface gravity waves on a fluid of arbitrary depth

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Nonlinear evolution equations (NEE's) are derived that describe weakly-two-dimensional motion of surface-gravity waves on a three-dimensional incompressible and inviscid fluid. When compared with the Kadomtsev-Petviashvili (KP) equation, the NEE's presented here have the following two advantages: (a) they are applicable to wave phenomena on a fluid of arbitrary depth; and (b) they can deal with head-on collisions of various wave structures. A discussion is then made of NEE's arising from the shallow- and deep-water limits. It is shown that the KP equation is a special case of our equations.

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Surface gravity waves in two spatial dimensions have been studied extensively over the past 150 years [1–7]. The main reason for this research is that the phenomena are familiar in nature and the mathematical problems involved are comparatively tractable. Various types of approximate nonlinear evolution equations (NEE's) have been derived that describe the wave motion on a fluid. A typical example is the famous Korteweg–de Vries (KdV) equation [2], which is the simplest unidirectional wave equation including the effects of nonlinearity and dispersion.

For the three-dimensional surface gravity waves, on the other hand, relatively little progress has been made due particularly to their inherently higher complexity. The satisfactory treatment of the fully three-dimensional problems has not been established even at the present time. However, if we restrict our consideration to weakly-two-dimensional wave motion by assuming that a wave field like the surface elevation varies slowly in the direction perpendicular to the propagation of the wave, we can obtain model NEE's which prove an accurate description of the time evolution of the wave. In the following discussion, we refer to the waves as one-dimensional or two-dimensional according to whether their surface profiles depend on one spatial variable or two spatial variables. Among several existing NEE's, the Kadomtsev-Petviashvili (KP) equation [8] arises as a natural generalization of the KdV equation. While the KP equation has been obtained in various areas of physical sciences and it has wide applications [9], the range of its validity is severely restricted by the following assumptions: The waves under consideration are (a) long compared with the depth of fluid, (b) undirectional, (c) small but finite in amplitude, and (d) weakly two dimensional. These must be supplemented by an *ad hoc* condition that the effects of (a), (c), and (d) balance in an appropriate manner. The assumption (a) is equivalent to the so-called shallow-water approximation. The KP equation has a very rich mathematical structure. Its initial-value problem has been solved via the inverse scattering transform [10]. Nevertheless, from the practical point of view, it is quite important to remove or to weaken these assumptions as far as possible.

The purpose of the present Brief Report is to eliminate (a) and (b) in the context of surface gravity waves and to derive NEE's which enable one to describe head-on collisions of various wave structures on a fluid of arbitrary depth. The method developed here is a generalization of a unified theory of surface gravity waves on a two-dimensional fluid of finite depth [6]. The assumption used throughout the analysis is that the wave steepness is small, in addition to weakly-two-dimensional motion [see (c) and (d) above]. A specific relation between nonlinearity and dispersion used in the KP theory is not introduced, as well as coordinate stretchings upon which most of the singular perturbation methods rely.

We consider the three-dimensional irrotational flow of an incompressible and inviscid fluid of uniform depth in a constant gravitational field. The effect of surface tension on the free surface is neglected to simplify the analysis, although it can be included without any difficulty. The dimensionless spatial and temporal coordinates are denoted by (x,y,z) and t, respectively. The gravitational acceleration g is in the negative z direction. The waves are assumed to be weakly two dimensional with the x direction dominant. In the dimensionless variables, the fluid motion is governed by the Laplace equation

$$\delta^2(\phi_{xx}+\mu\phi_{yy})+\phi_{zz}=0 \qquad (-\infty<x,y<\infty,\ -1<z<\alpha\eta)\ , \tag{1}$$

with the kinematic and dynamical conditions on the free surface

$$\eta_t+\kappa\epsilon(\phi_x\eta_x+\mu\phi_y\eta_y)=\frac{\kappa}{\delta}\phi_z \quad (z=\alpha\eta)\ , \tag{2}$$

$$\phi_t+\frac{\kappa\epsilon}{2\delta^2}[\delta^2(\phi_x^2+\mu\phi_y^2)+\phi_z^2]+\eta-\eta_0=0 \quad (z=\alpha\eta)\ , \tag{3}$$

and the boundary condition on the bottom of fluid

$$\phi_z=0 \quad (z=-1)\ . \tag{4}$$

Here $\phi=\phi(x,y,z,t)$ is the velocity potential, $\eta=\eta(x,y,t)$ is the free-surface evaluation, and the subscripts x,y,z, and t appended to ϕ and η denote partial differentiations. η is assumed to be negative and the origin of the coordinate system is chosen such that $\max\eta=0$. η_0 is a constant determined by the boundary conditions for ϕ and η

at infinity. The dimensional quantities, with tildes, are related to the corresponding dimensionless ones by the relations $\tilde{x}=lx$, $\tilde{y}=l'y$, $\tilde{z}=h_0z$, $\tilde{t}=(l/c_0)t$, $\tilde{\phi}=(gla/c_0)\phi$, and $\tilde{\eta}=a\eta$, where l and l' are characteristic length scales of the wave for the x and y directions, respectively, a is a typical amplitude of the wave, and c_0 is the phase velocity of the wave. The bottom topography of fluid is represented by the plane $z=-h_0$. The dimensionless parameters α, δ, ϵ, and μ are defined according to the relations

$$\alpha=a/h_0\ ,\quad \delta=h_0/l\ ,\quad \epsilon=a/l\ ,\quad \mu=(l/l')^2\ . \tag{5}$$

In (5) ϵ is the steepness parameter which measures the magnitude of nonlinearity. A useful relation $\epsilon=\alpha\delta$ should be remarked upon. c_0 is given by $c_0=\sqrt{gl/\kappa}$, where κ depends on δ with asymptotic values δ^{-1} in the shallow-water limit $\delta\rightarrow 0$ and 1 in the deep-water limit $\delta\rightarrow\infty$, in accordance with the phase velocity of linear surface gravity waves. In contrast to the two-dimensional case, there appears a new parameter μ which can be taken arbitrarily insofar as it is small compared with unity because of the assumption of weak two-dimensionality. In the present study, we specify μ such that

$$\mu=\nu\alpha\ , \tag{6}$$

where ν is a numerical constant of ~ 1. The reason for this choice is that it leads correctly to the KP equation in the shallow-water limit, as will be shown below. Although another scaling is possible for μ which would give rise to another type of NEE's, we shall not discuss it here and focus our attention to the case (6).

Let us now derive approximate NEE's correct up to $O(\epsilon)$ by means of a systematic perturbation method with respect to ϵ. The extension of NEE's to higher orders can be made straightforwardly, and hence we shall omit all details. We first take the solution of (1) satisfying the bottom boundary condition (4) of the form

$$\phi=i\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{sgn}(k_x)\cosh[\delta k(z+1)]}{\sinh(\delta k)}\hat{f}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\ , \tag{7}$$

where the two-dimensional vectors $\mathbf{x}$ and $\mathbf{k}$ are defined, respectively, by $\mathbf{x}=(x,y)$ and $\mathbf{k}=(k_x,k_y)$, $\hat{f}(\mathbf{k},t)$ is an arbitrary function, $k=(k_x^2+\mu k_y^2)^{1/2}$, $\mathrm{sgn}(k_x)$ is the sign function, and the symbol P denotes the Cauchy principal-value integral. The next step is to expand the integrand in powers of μ [or ϵ by virtue of the relations (6) and $\epsilon=\alpha\delta$] and then rewrite the resultant expression in terms of physical variables by employing a technique of the Fourier transform. In so doing, it is important to recall that the depth parameter δ is a quantity of ~ 1. After some calculations, we find that the desired expression can be written in the form

$$\begin{aligned}\phi=-i(1+(i\nu\epsilon/2)\{&(z+1)\tanh[i\delta(z+1)\partial_x]\\&-\coth(i\delta\partial_x)\}\partial_x^{-1}\partial_y^2+O(\epsilon^2))\\&\times[f_+(x-i\delta z,y,t)-f_-(x+i\delta z,y,t)]\ .\end{aligned} \tag{8}$$

Here $f_+(\zeta,y,t)$ $[f_-(\zeta,y,t)]$ is an analytic function of ζ in the strip $0<\mathrm{Im}\zeta<2\delta$ $(-2\delta<\mathrm{Im}\zeta<0)$ and given explicitly by

$$f_\pm(\zeta,y,t)=\pm\frac{1}{4i\delta}\int_{-\infty}^{\infty}\coth[\pi(x'-\zeta)/2\delta]f(x',y,t)dx'\ , \tag{9}$$

with $f(x,y,t)$ being the Fourier transform of $\hat{f}$, i.e., $f(\mathbf{x},t)=\int_{-\infty}^{\infty}\hat{f}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}$. The notations $\partial_x f=\partial f/\partial x$, $\partial_y^2 f=\partial^2 f/\partial y^2$, and $\partial_x^{-1}f=\int^x f(x')dx'$ have been used together with the formulas

$$\begin{aligned}\coth(i\delta\partial_x)e^{ik_x x}&=-\coth(\delta k_x)e^{ik_x x}\ ,\\ \tanh(i\delta\partial_x)e^{ik_x x}&=-\tanh(\delta k_x)e^{ik_x x}\ .\end{aligned} \tag{10a}$$

It is worthwhile to note that the Fourier transform of these formulas yield the corresponding representations expressed in physical variables as

$$\coth(i\delta\partial_x)f=iTf\ ,\quad \tanh(i\delta\partial_x)f=-i\tilde{T}f\ , \tag{10b}$$

where the singular integral operator T and its inverse $\tilde{T}$ acting only on the x variable are defined by

$$Tf(x)=\frac{1}{2\delta}\mathrm{P}\int_{-\infty}^{\infty}\coth[\pi(x'-x)/2\delta]f(x')dx'\ , \tag{11a}$$

$$\tilde{T}f(x)=-\frac{1}{2\delta}\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x')}{\sinh[\pi(x'-x)/2\delta]}dx'\ . \tag{11b}$$

The solution (8) is the main ingredient in this article and becomes the starting point in the following perturbation analysis. If the y dependence is neglected, it coincides perfectly with the corresponding solution for the two-dimensional velocity potential obtained in [6].

Now in order to derive an approximate system of NEE's from (2) and (3), it is necessary to evaluate the derivatives of the velocity potential on the free surface. This can be done by expanding $f_\pm$ in ϵ and invoking the important relations

$$f_+(x+i0,y,t)+f_-(x-i0,y,t)=f(x,y,t)\ , \tag{12}$$

$$f_+(x+i0,y,t)-f_-(x-i0,y,t)=-iTf(x,y,t)\ , \tag{13}$$

which come from (9) with the help of the residue theorem. The expressions for the first two terms of the expansions thus obtained read in the forms

$$\phi_x|_{z=\alpha\eta}=-Tf_x-\epsilon\eta f_{xx}-\frac{\nu\epsilon}{2}(1+T^2)f_{yy}+O(\epsilon^2)\ , \tag{14}$$

$$\phi_z|_{z=\alpha\eta}=-\delta(f_x-\epsilon\eta Tf_{xx})-\frac{\nu\epsilon}{2}\partial_x^{-1}f_{yy}+O(\epsilon^2)\ , \tag{15}$$

$$\phi_t|_{z=\alpha\eta}=-Tf_t-\epsilon\eta f_{xt}-\frac{\nu\epsilon}{2}\partial_x^{-1}(1+T^2)f_{yyt}+O(\epsilon^2)\ . \tag{16}$$

At this stage we introduce the x component of the surface velocity by $u\equiv\phi_x|_{z=\alpha\eta}$. Then the quantity f_x in (14) can be solved iteratively in terms of u and η as

$$f_x=-\tilde{T}u+\epsilon[\tilde{T}(\eta\tilde{T}u_x)+(\nu/2)\partial_x^{-1}(1+\tilde{T}^2)u_{yy}]+O(\epsilon^2)\ . \tag{17}$$

Substitution of (17) into the x derivatives of (15) and (16) yields

$$(\phi_z|_{z=\alpha\eta})_{xx} = -\delta[-\tilde{T}u_{xx} + \epsilon\{\tilde{T}(\eta\tilde{T}u_x)_{xx} + (\eta u_x)_{xx} - (\nu/2\delta)\tilde{T}u_{yy} + (\nu/2)(1+\tilde{T}^2)u_{xyy}\} + O(\epsilon^2)] , \quad (18)$$

$$(\phi_t|_{z=\alpha\eta})_x = u_t + \epsilon(\eta_x\tilde{T}u_t - \eta_t\tilde{T}u_x) + O(\epsilon^2) . \quad (19)$$

The time evolutions of η and u follow immediately by introducing (18) and (19) into the x derivatives of (2) and (3) and retaining the terms of $O(\epsilon)$. We quote only the final result as follows:

$$\eta_{xxt} - \kappa\tilde{T}u_{xx} + \kappa\epsilon[(u\eta)_{xxx} + \tilde{T}(\eta\tilde{T}u_x)_{xx} - (\nu/2\delta)\tilde{T}u_{yy} + (\nu/2)(1+\tilde{T}^2)u_{xyy}] + O(\epsilon^2) = 0 , \quad (20)$$

$$u_t + \eta_x + \epsilon(\kappa u u_x - \eta_x\tilde{T}\eta_x) + O(\epsilon^2) = 0 . \quad (21)$$

The above NEE's constitute a closed system and hence they can be solved under appropriate initial and boundary conditions. However, for practical purpose, it is sometimes useful to derive a single equation for η, which we shall now carry out. We first solve (20) with respect to κu_x and express it in terms of η to obtain

$$\kappa u_x = T\eta_{xt} + \epsilon[T(\eta T\eta_t)_{xx} + (\eta\eta_{xt})_x + (\nu/2)(1+T^2)\eta_{yyt} - (\nu/2\delta)\partial_x^{-1}T\eta_{yyt}] + O(\epsilon^2) , \quad (22)$$

where use has been made of the lowest-order equation $\kappa u = T\eta_t + O(\epsilon)$ in $O(\epsilon)$ terms to eliminate u. Finally, substituting (22) into the x derivative of (21) and then operating $\tilde{T}$ on the resultant equation, we find the time evolution of η as follows:

$$\eta_{xtt} + \kappa\tilde{T}\eta_{xx} + \epsilon[\{-\kappa\eta\eta_x + 2\eta_t T\eta_t + \tilde{T}\eta_t^2 - \kappa\tilde{T}(\eta\tilde{T}\eta_x)\}_{xx} + (\kappa\nu/2\delta)\tilde{T}\eta_{yy} - (\kappa\nu/2)(1+\tilde{T}^2)\eta_{xyy}] + O(\epsilon^2) = 0 . \quad (23)$$

Here in $O(\epsilon)$ terms, we have replaced η_{tt} by the lowest-order equation $\eta_{tt} = -\kappa\tilde{T}\eta_x + O(\epsilon)$ and used the formula $\tilde{T}(fg) = \tilde{T}[(\tilde{T}f)(\tilde{T}g)] + g\tilde{T}f + f\tilde{T}g$.

The NEE's (20), (21), and (23) are main results in this Brief Report. If the y dependence is neglected, these equations reduce to the corresponding NEE's for the one-dimensional wave [6]. It is easy to show with the aid of (10) that a linearized version of (23)

$$\eta_{xtt} + \kappa\tilde{T}\eta_{xx} + \epsilon\left[\frac{\kappa\nu}{2\delta}\tilde{T}\eta_{yy} - \frac{\kappa\nu}{2}(1+\tilde{T}^2)\eta_{xyy}\right] = 0 , \quad (24)$$

reproduces the linear dispersion relation relevant to the present problem as

$$\omega^2 = \kappa k_x \tanh(\delta k_x) + \frac{\kappa\epsilon}{2\delta}\left[\frac{k_y^2}{k_x}\tanh(\delta k_x) + \delta k_y^2 \operatorname{sech}^2(\delta k_x)\right] . \quad (25)$$

Lastly, we shall comment on the sign of η, which has been assumed to be negative in the present analysis. We show that it can be taken to be positive without changing the NEE's themselves as follows: We first shift the origin of the z coordinate downward by a and replace η by $-1+\eta$. At the same time, it is suitable to replace δ by $\delta+\epsilon$; thereby the undisturbed depth of fluid becomes h_0. After that we expand the operators T and $\tilde{T}$ in powers of ϵ to obtain the corresponding replacements

$$T_f \to T_f + \epsilon(1+T^2)f_x + O(\epsilon^2) ,$$
$$\tilde{T}f \to \tilde{T}f - \epsilon(1+\tilde{T}^2)f_x + O(\epsilon^2) .$$

Completing all the above procedures in (20), (21), and (23), we find that the transformed NEE's take exactly the same forms as those of the original ones.

The NEE's obtained here are uniformly valid from shallow water to deep water and have wide applications in various wave phenomena of physical and engineering importance. In specific situations, however, it is simpler to employ NEE's arising from our equations as the limiting cases of fluid depth. Therefore we next consider the shallow- and deep-water limits of the equations and discuss the property of the resulting NEE's.

(i) Shallow-water limit ($\delta \to 0$). In this limit we take $\kappa = \delta^{-1}$ and specify the relative magnitude of the parameters α and δ as $\alpha = O(\delta^2)$, which is consistent with the fundamental assumption in the derivation of the KP equation [8]. If we employ the small δ expansions for the operators $Tf = (1/2\delta)\int_{-\infty}^{\infty}\operatorname{sgn}(x'-x)f(x')dx' + O(\delta)$ and $\tilde{T}f = -\delta f_x - (\delta^3/3)f_{xxx} + O(\delta^5)$, the NEE's (20), (21), and (23) reduce, respectively, to

$$\eta_{xt} + u_{xx} + (\delta^2/3)u_{xxxx} + \alpha[(u\eta)_{xx} + \nu u_{yy}] + O(\alpha\delta^2, \delta^4) = 0 , \quad (26)$$

$$u_t + \eta_x + \alpha u u_x + O(\alpha\delta^2) = 0 , \quad (27)$$

$$\eta_{tt} - \eta_{xx} - (\delta^2/3)\eta_{xxxx} + \alpha\left[\left[-\eta\eta_x + \eta_t\int_{-\infty}^{\infty}\operatorname{sgn}(x'-x)\eta_t(x',y,t)dx'\right]_x - \nu\eta_{yy}\right] + O(\alpha\delta^2, \delta^4) = 0 . \quad (28)$$

Equation (28) is fit to describe weakly nonlinear long waves propagating to both right and left directions and hence it can deal with the interaction between these waves as well. It is interesting to note that (28) exhibits a solitary wave solution of the form

$$\eta = A\,\mathrm{sech}^2(k_x x + k_y y - \omega t)\ , \tag{29}$$

with $A = 4\delta^2 k_x^4/[\alpha(2\omega^2+k_x^2)]$ and $\omega^2 = k_x^2 + 4\delta^2 k_x^4/3 + \alpha\nu k_y^2$, where we have assumed positive η and vanishing boundary value [see the comment following (25)]. To obtain a NEE describing a unidirectional motion to the right, for instance, we follow the standard procedure [3]. It then turns out from (28) that the right running waves evolve according to the KP equation [8]

$$[\eta_t + \eta_x + \tfrac{3}{2}\alpha\eta\eta_x + (\delta^2/6)\eta_{xxx}]_x + (\nu\alpha/2)\eta_{yy} + O(\alpha\delta^2,\delta^4) = 0\ . \tag{30}$$

(ii) Deep-water limit ($\delta \to \infty$). In this limit it is appropriate to choose $\kappa = 1$ and further to rescale the y and z coordinates as $y = \hat{y}/\sqrt{\delta}$ and $z = \hat{z}/\delta$, respectively. The variables $\hat{y}$ and $\hat{z}$ are then related to the original physical variables by $\tilde{y} = (l/\sqrt{\epsilon\nu})\hat{y}$ and $\tilde{z} = l\hat{z}$. Consequently, in the limit of $\delta \to \infty$, there exists only the one small parameter ϵ in the system under consideration. Since the operators T and $\tilde{T}$ reduce to H and $-H$, respectively, where H is the Hilbert transform given by $Hf = (1/\pi)\mathrm{P}\int_{-\infty}^{\infty}(x'-x)^{-1}f(x')dx'$, (20), (21), and (23) become simply to the following NEE's:

$$\eta_{xxt} + Hu_{xx} + \epsilon[(u\eta)_{xx} + H(\eta H u_x)_{xx} + (\nu/2)Hu_{\hat{y}\hat{y}}] + O(\epsilon^2) = 0\ , \tag{31}$$

$$u_t + \eta_x + \epsilon(uu_x + \eta_x H\eta_x) + O(\epsilon^2) = 0\ , \tag{32}$$

$$\eta_{xtt} - H\eta_{xx} + \epsilon[\{-\eta\eta_x + 2\eta_t H\eta_t - H\eta_t^2 - H(\eta H\eta_x)\}_{xx} - (\nu/2)H\eta_{\hat{y}\hat{y}}] + O(\epsilon^2) = 0\ . \tag{33}$$

In this paper we have presented an alternative type of NEE's for weakly-two-dimensional motion of surface gravity waves on a fluid of arbitrary depth and demonstrated that the KP equation stems naturally from our system. Although the NEE's obtained here are still insufficient for the purpose of the description of genuinely-two-dimensional wave motion, these have a greater flexibility in applications in real physical systems when compared with the applicability of the KP equation.

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